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Application of the Principle of the Minimum Maximum Modulus to Generalized Moment Problems and Some Remarks on Quantum Field Theory*

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We develop a method for separating into the component parts functions which are the difference of two series of Stieltjes, and with other related but more complex structures which occur in certain physical problems. Our approach differs from the classical one in that we also can construct a convergent sequence of values to the function value throughout the cut complex plane.

1. INTRODUCTION AND SUMMARY

Padé approximants [1, 2] provide a powerful means of extrapolating and interpolating functions defined by their Taylor series, and in some cases by their values (and derivatives) at several points. When the function to be studied is a member of the class of series of Stieltjes then the Padé approximants actually provide convergent bounds on the value throughout the cut complex z-plane [2, 3]. A series of Stieltjes is a function of the form

$$f(z) = \int_0^\infty \frac{d\varphi(t)}{1 + tz} \tag{1.1}$$

where $d\varphi \geqslant 0$. It is plainly of value to extend the class of functions to which this type of proven convergence applies.

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There are various physical problems for which the function to be studied has the form of a simple combination of more than one series of Stieltjes, e.g. the difference of two such [4]. If we could untangle these series, then we could estimate their values separately in a convergent way and hence obtain a set of approximations to the function which is known to be convergent. Another example in which we are interested, because of closely similar problems which arise in the fundamental structure of field theory, comes from potential scattering theory. It turns out, for suitably restricted potentials, that the tangent of the scattering phase shift is of the form,

$$\tan \delta(E) = \frac{U(E) - V(E)}{1 + EU(E)\,V(E)}\,, \tag{1.2}$$

where $U(E)$ and $V(E)$ are series of Stieltjes in $(-E)$. This assertion is proved in Appendix A. The closely similar problems which arise in field theory are discussed in Appendix B.

The classical problem of separating the difference of two series of Stieltjes has been solved by Reisz and Hausdorff [5]. The trouble with their sequence of approximations is that although it tends to a series of Stieltjes in the limit, the individual approximations (in the form of finite numbers of coefficients) are not and so we cannot use them to form convergent approximations to the function values throughout the cut z-plane.

To overcome this handicap we develop in the second section a method based on the principle of the minimum maximum modulus which allows us to construct a convergent sequence of series of Stieltjes. We then generalize our procedures to cover the case where the radius of convergence is zero. Here it is necessary to specify the functional class in which we work to insure uniqueness. In the third section we illustrate for functions of type (1.2) how our methods can be adapted to treat more general problems.

2. The Moment Problem for a Measure of Bounded Variation

We will deal first with the problem with a finite radius of convergence. For convenience we will use unity for this radius. Suppose we have a series

$$C(x) = \sum_{p=0}^{\infty} C_p(-x)^p \tag{2.1}$$

such that

$$C_p = \int_0^1 t^p \, d\psi(t) \tag{2.2}$$

where ψ is restricted so that

$$\int_0^1 | d\psi(t)| < M \tag{2.3}$$

that is, ψ is of bounded variation. The function ψ (if it exists) is substantially unique [6]. We seek to reexpress $C(x)$ in terms of $A(x)$ and $B(x)$ such that

$$C(x) = A(x) - B(x) \tag{2.4}$$

where

$$A(x) = \int_0^1 \frac{d\psi_A(u)}{1 + ux} \tag{2.5}$$

$$B(x) = \int_0^1 \frac{d\psi_B(u)}{1 + ux} \tag{2.6}$$

where $d\psi_A \geqslant 0$ $d\psi_B \geqslant 0$. We can then assuredly use Padé approximants to provide proven convergent estimates to $C(x)$ of this form [1].

Let us consider the special solutions

$$d\hat{\psi}_A(u) = U(d\psi(u)) \tag{2.7}$$

$$d\hat{\psi}_B(u) = U(-\, d\psi(u)), \tag{2.8}$$

where

$$\begin{aligned} U(x) &= x, \qquad x > 0 \\ U(x) &= 0, \qquad x \leqslant 0. \end{aligned} \tag{2.9}$$

Now substituting $d\hat{\psi}_A$ and $d\hat{\psi}_B$ in (2.5–6) gives a representation of (2.1) via (2.4) as desired. Furthermore as $(d\hat{\psi}_A)\,(d\hat{\psi}_B) = 0$, i.e. ψ cannot be positive and negative at the same time, so that the support S_A and S_B of the distributions $d\hat{\psi}_A$ and $d\hat{\psi}_B$ is disjoint, i.e. $S_A \cap S_B = \varnothing$.

Any solution to this problem can be written in the form

$$\begin{aligned} d\psi_A &= d\hat{\psi}_A + d\omega \\ d\psi_B &= d\hat{\psi}_B + d\omega, \end{aligned} \tag{2.10}$$

as we must have

$$d\psi = d\psi_A - d\psi_B = d\hat{\psi}_A - d\hat{\psi}_B\,, \tag{2.11}$$

since $d\psi$ is substantially unique. Now as we require *both* $d\psi_A$ and $d\psi_B$ to be series of Stieltjes ($d\psi_A > 0$, $d\psi_B > 0$) we must have $d\omega \geqslant 0$. For as the region where $d\hat{\psi}_A$ or $d\hat{\psi}_B = 0$ covers the unit interval

$$((I - S_A) \cup (I - S_B)) = I - S_A \cap S_B = I - \varnothing = I, \tag{2.12}$$

a negative value of $d\omega$ would make either $d\psi_A$ or $d\psi_B$ negative and so not a series of Stieltjes.

Now let us consider, over a circle of radius $r < 1$,

$$\max_{|z|\leqslant r} \{| A(z)|\} \qquad \text{and} \qquad \max_{|z|\leqslant r} \{| B(z)|\}. \tag{2.13}$$

Since A and B are series of Stieltjes their coefficients alternate in sign. Thus the maximum modulus occurs at $z = -r$ for $|z| \leqslant r$. This result is true both of the complete function and a truncated series expansion. Let us then consider

$$\min_{\{d\psi_A, d\psi_B\}} [\max_{|z|\leqslant r} \{| A(z)|\} + \max_{|z|\leqslant r} \{| B(z)|\}]$$

$$= \min_{\{d\psi_A, d\psi_B\}} \int_0^1 \frac{d\psi_A + d\psi_B}{1 - ru} \tag{2.14}$$

$$= \min_{\{d\omega\}} \left\{ \int_0^1 \frac{d\hat{\psi}_A + d\hat{\psi}_B + 2d\omega}{1 - ru} \right\}. \tag{2.15}$$

As $d\hat{\psi}_A$ and $d\hat{\psi}_B$ are fixed quantities. This problem is equivalent to

$$\min_{\{d\omega\}} \int_0^1 \frac{2d\omega}{1 - ru}. \tag{2.16}$$

The solution to this problem is obvious as $1/(1 - ru) > 0$ for all $0 \leqslant u \leqslant 1$ and $d\omega \geqslant 0$. Thus $d\omega = 0$ and the solution to the following problem can be given essentially uniquely.

The $A(x)$ and $B(x)$ which yield a $C(x)$ of the form (2.1, 2) according to (2.4–6) subject to the condition (2.14) is uniquely the particular solution given by measures (2.7) and (2.8). This solution is independent of the value r, for $0 < r < 1$.

The prescription then to solve our problem is first to determine a sequence of approximations indexed by N such that

$$\begin{aligned} D_A(0, n) &> 0, \qquad D_B(0, n) > 0 \\ D_A(1, n) &> 0, \qquad D_B(1, n) > 0, \end{aligned} \tag{2.17}$$

where

$$D_A(m, n) = \det \begin{vmatrix} a_m & a_{m+1} & \cdots & a_{m+n} \\ a_{m+1} & a_{m+2} & \cdots & a_{m+n+1} \\ \vdots & & & \\ a_{m+n} & a_{m+n+1} & \cdots & a_{m+2n} \end{vmatrix}, \tag{2.18}$$

and similarly for B series

$$A(x) = \sum_{p=0}^{\infty} a_p(-x)^p, \qquad B(x) = \sum_{p=0}^{\infty} b_p(-x)^p, \tag{2.19}$$

for all values of n for which

$$m + 2n \leqslant N.$$

(These conditions are necessary and sufficient [1] for there to exist a non-negative definite measure function for $A(x)$ and for $B(x)$ which will reproduce the coefficients $a_0, a_1, ..., a_N$ and $b_0, b_1, ..., b_N$). We further require

$$a_j - b_j = c_j, \qquad j = 0, 1, ..., N. \tag{2.20}$$

From among the infinity of such functions we select for our Nth approximation, that one for which

$$\sum_{j=0}^{N} (a_j^{(N)} + b_j^{(N)})\, r^j \tag{2.21}$$

is a minimum. This minimum maximum modulus is bounded from above (from (2.3)) by

$$\min \sum_{j=0}^{N} (a_j^{(N)} + b_j^{(N)})\, r^j \leqslant \frac{M}{1-r} \tag{2.22}$$

as the known to exist solutions satisfy all our conditions except for (2.21), and is bounded by (2.22). Furthermore the first N terms of the $(N+1)$st approximation must satisfy

$$\sum_{j=0}^{N} (a_j^{(N)} + b_j^{(N)})\, r^j \leqslant \sum_{j=0}^{N} (a_j^{(N+1)} + b_j^{(N+1)})\, r^j. \tag{2.23}$$

As the first is a minimum and therefore, *a fortiori*,

$$\leqslant \sum_{j=0}^{N+1} (a_j^{(N+1)} + b_j^{(N+1)})\, r^j. \tag{2.24}$$

Hence the minimum maximum modulus increases, monotonically and is bounded from above.

Thus

$$\lim_{N\to\infty} [A^{(N)}(-r) + B^{(N)}(-r)] \tag{2.25}$$

exists. As $A^{(N)}(-r) - B^{(N)}(-r)$ converges therefore $A^{(N)}(-r)$ and $B^{(N)}(-r)$ separately converge to a limit. Now since (2.22) the minimum maximum modulus is bounded and is a sum of positive terms, each term is bounded, therefore

$$0 \leqslant a_j^{(N)} + b_j^{(N)} \leqslant \frac{Mr^{-j}}{1-r} \tag{2.26}$$

for all (N) and j. Hence by well known theorems there exists an infinite subsequence of (N)'s for which $a_0^{(N)} + b_0^{(N)}$ tends to a limit. From this subsequence we can select an infinite subsequence for which both $a_0^{(N)} + b_0^{(N)}$ and $a_1^{(N)} + b_1^{(N)}$ converge etc. Thus there exists at least a subsequence which converges to a limit. But our conditions determine the limiting function substantially uniquely. Hence, the whole sequence must converge; for if there were an infinite number of values further than $\epsilon > 0$ from the correct limit we could select a convergent subsequence from them which would yield a solution to the problem which differs by at least ϵ. But this situation is impossible as the solution is unique.

Therefore we conclude that our sequence of approximants converges and it converges to the solution with disjoint measure functions. The resultant limit function is independent of the assumed value of r provided $0 < r < 1$.

If we desire the function value at $z = -r$, then the maximum moduli of A and B converge monotonically to $A(-r)$ and $B(-r)$.

Let us now consider the case where the radius of convergence is zero. Suppose we are now given a series

$$C(x) = \sum_{p=0}^{\infty} C_p(-x)^p \tag{2.27}$$

such that

$$C_p = \int_0^{\infty} t^p \, d\psi(t), \tag{2.28}$$

where we restrict ψ so that d_p defined by

$$d_p = \int_0^{\infty} t^p \mid d\psi(t) \mid \tag{2.29}$$

has the property

$$\sum_{p=0}^{\infty} (d_p)^{-1/(2p)} \text{ diverges.} \tag{2.30}$$

This property [1] is sufficient to ensure that the absolute moments uniquely determine a $\mid d\psi \mid$. [We note that condition (2.3) implies that all the absolute moments for the radius of convergence $= 1$ case are bounded by M.]

We now establish, within this class, that the ψ corresponding to the C_p is essentially unique. Suppose there exist 2 solutions ψ_1 and ψ_2, each of the class defined by (2.29). We may write

$$\begin{aligned} \psi_1(t) &= \psi_1'(t) - \psi_1''(t), \\ \psi_2(t) &= \psi_2'(t) - \psi_2''(t), \end{aligned} \tag{2.31}$$

where ψ_1', ψ_1'', ψ_2', ψ_2'' are all bounded non-decreasing functions of t. Our assumption yields for $n = 0, 1, 2,\ldots$

$$\int_0^\infty t^n[d(\psi_1'(t) + \psi_2''(t))] = \int_0^\infty t^n\, d(\psi_2'(t) + \psi_1''(t)). \tag{2.32}$$

Since the right and left sides are equal and positive, each is equal to one half the sum of both sides

$$= \tfrac{1}{2}\,(d_n^{(1)} + d_n^{(2)}). \tag{2.33}$$

But as $\sum\, [d_p^{(i)}]^{-1/(2p)}$ for $i = 1$ or 2 diverges by assumption, so does

$$\sum_{p=0}^{\infty}\, [d_p^{(1)} + d_p^{(2)}]^{-1/(2p)}. \tag{2.34}$$

Thus we have two bounded non-decreasing functions, $\psi_1'(t) + \psi_2''(t)$ and $\psi_2'(t) + \psi_1''(t)$, which have the same moments to all orders over the interval $(0, \infty)$. Furthermore, the moments are of the determinate type [6]. Thus these two functions are substantially equal. Hence ψ_1 is substantially equal to ψ_2 and the solution within this class is substantially unique. Note this result is in contrast to the unrestricted result of Boas [7] and Polya [8] where they show that there exist infinitely many ψ's of bounded variation for which the same moments result.

Now the prescription we gave for radius of convergence $= 1$ type series does not discriminate against radius of convergence $= 0$, except for the principle of the minimum maximum modulus which requires a radius of convergence $\geqslant r$. Hence we propose a new maximum modulus function

$$\sum_{j=0}^{N} (a_j^{(N)} + b_j^{(N)}) \frac{r^j}{\Gamma(\theta j + 1)} \tag{2.35}$$

where $0 \leqslant \theta \leqslant 2$, and $\Gamma(x)$ is Legendre's gamma function. This function has the representation

$$\sum_{j=0}^{\infty} \alpha_j \frac{r^j}{\Gamma(\theta j + 1)} = \int_0^\infty \left\{ \sum_{j=0}^{\infty} \frac{(rt)^j}{\Gamma(\theta j + 1)} \right\} d\psi_\alpha(t) \tag{2.36}$$

provided the left hand side series converges. Suppose

$$\alpha_j \leqslant K\Gamma(\theta j + 1)\, R^{-j}. \tag{2.37}$$

Then

$$\sum_{j=0}^{\infty} \frac{\alpha_j r^j}{\Gamma(\theta j + 1)} \leqslant \frac{K}{1 - r/R}, \tag{2.38}$$

which is a bound as long as $r < R$. Also if θ in (2.35) is greater than θ in (2.37) we have a bound for arbitrary positive, finite r and R. Hence as { } in (2.36) is positive, [as in (2.16) $(1 - ru)^{-1}$ was] the derivation follows as before (radius of convergence $= 1$ case). If we minimize the criterion (2.35) [r and θ selected such that

$$\left.\sum_{p=0}^{\infty} \frac{d_p r^p}{\Gamma(\theta j + 1)} < M\right], \tag{2.39}$$

then there must be at least an infinite subsequence of our approximations which converges, and any convergent subsequence must converge to the A and B corresponding to the substantially unique, disjoint, measure functions $\hat{\psi}_A$ and $\hat{\psi}_B$. This result is independent of r and θ provided (2.39) converges.

As a practical matter one might think one should always just use $1/(2n)!$ as a weight to do the widest possible class, but we presume a more rapidly convergent sequence results if the criterion which most nearly matches the actual answer is used.

3. The Moment Problem for More General Functional Types

The procedures of Section 2 can also be applied to more general types of problems. We illustrate this with

$$f(z) = \frac{U(z) - V(z)}{1 + zU(z)\, V(z)} \tag{3.1}$$

or

$$V(z) = \frac{U(z) - f(z)}{1 + zU(z) f(z)} \tag{3.2}$$

where it is understood that $U(z)$, and $V(z)$ are sought, and they are required to be series of Stieltjes and $f(z)$ is given to be of form (3.1) for some U_1 and V_1. Plainly, if there is known to be a solution then it will provide an upper bound for (2.21) or (2.35). Hence the procedure described must again have a convergent subsequence which yields a solution of minimum modulus. Furthermore, if either U_1 or V_1 is zero, then the solution is unique, independent of the choice of subsequences, r, and θ, as before. If $U_1 V_1 \not\equiv 0$, then we have not proved any of the uniqueness properties which were available in the previous section. However, any solution will correspond to the unique $f(z)$ and as estimates for U and V separately tend to have errors related to their size minimizing $\{\max U + \max V\}$ will, as a practical matter tend to improve the quality of the (*known to be convergent*) estimates of f throughout the cut complex z-plane.

APPENDIX A. THE STRUCTURE OF $\tan[\delta(E)]$

A proof that $\tan[\delta(E)]$ has the structure asserted in Section 1 can be given as follows. For potential scattering (angular momentum zero, but this restriction is of no consequence) the wave function satisfies Schrödinger's equation [9],

$$\frac{d^2\psi(r)}{dr^2} + [E - V(r)]\,\psi(r) = 0. \tag{A.1}$$

We will assume that $V(r)$ is such that

$$V(r) = 0, \qquad r \geqslant c \tag{A 2}$$

for some $0 < c < \infty$. We now introduce a complete, orthonormal (over the range $0 \leqslant r \leqslant c$) set of wave functions defined by the requirements

$$\begin{gathered} \frac{d^2\varphi_n(r)}{dr^2} + [k_n{}^2 - V(r)]\,\varphi_n(r) = 0, \\ \varphi_n(0) = 0, \qquad \varphi_n{}'(c) = 0, \end{gathered} \tag{A.3}$$

which determines both φ_n and $k_n{}^2$. Let us now expand the solution of (A.1) as

$$\psi(r) = \sum_n A_n \varphi_n(r). \tag{A.4}$$

We remark that at $r = c$, Eq. (A.4) is not differentiable, but we must use $\psi'(c) = \lim_{r\to c-} \psi'(r)$ instead. The coefficients are given in the standard way as

$$A_n = \int_0^c \varphi_n(r)\,\psi(r)\,dr \tag{A.5}$$

where ψ is the solution of (A.1) for general E. If we multiply (A.3) by $\psi(r)$, subtract $\varphi_n(r)$ times (A.1), and integrate from 0 to c, we obtain

$$\begin{aligned}\int_0^c (k_n^2 - E)\,\varphi_n(r)\,\psi(r)\,dr &= [\psi'(r)\,\varphi_n(r) - \varphi_n'(r)\,\psi(r)]\,|_0^c \\ &= \varphi_n(c)\,\psi'(c),\end{aligned} \tag{A.6}$$

or using (A.5) we have

$$A_n = \frac{1}{k_n^2 - E}\,\varphi_n(c)\,\psi'(c). \tag{A.7}$$

Thus

$$\psi(r) = \sum_n \frac{1}{k_n^2 - E}\,\varphi_n(c)\,\psi'(c)\,\varphi_n(r). \tag{A.8}$$

If we evaluate (A.8) for $r = c$ we obtain

$$D \equiv \frac{\psi(c)}{\psi'(c)} = \sum_n \frac{\varphi_n^2(c)}{k_n^2 - E}, \tag{A.9}$$

which is a series of Stieltjes in $(-E)$ provided, $k_n^2 > 0$ for all n. (This condition means that V produces no bound states.) Using the standard method of matching logarithmic derivatives at $r = c$ to determine the phase shift, we obtain

$$\tan[\delta(E)] = \frac{D - [\tan(\sqrt{E}c)/(\sqrt{E}c)}{1 + E[D]\,[\tan(\sqrt{E}c)/(\sqrt{E}c)]}. \tag{A.10}$$

Now as $[\tan(\sqrt{E}c)/(\sqrt{E}c)]$ is a known series of Stieltjes [1], (A.10) establishes that $\tan[\delta(E)]$ has the required form.

APPENDIX B. Remarks on Quantum Field Theory

One may ask why it is of interest to connect the renormalized series which result from field theory to series of Stieltjes. Our answer depends on concepts given by Carleman [10]. A function belongs to the class $(A_0, A_1, \ldots)$ provided that its derivatives satisfy

$$|f^{(n)}(x)| \leqslant A_n \tag{B.1}$$

on some interval containing the origin (the origin may be a boundary of the interval). Provided that

$$\sum \left(\frac{1}{A_n^{1/n}}\right) \text{ diverges,} \tag{B.2}$$

there exists at most one function belonging to class $(A_0, A_1, \ldots)$ and having prescribed derivatives at the origin [11]. Now the bounds on the derivatives provided by Eq. (B.1) are equivalent to the statement that $f(x)$ is smooth. It would be nice to know that the sum of a renormalized series resulting from field theory is unique subject only to the condition that it be smooth.

However, the theorem quoted does not suffice, because it is thought that the terms c_n $(c_n n! = f^{(n)}(0))$ in field theory increase like $(2n)!$, for which the sum in Eq. (B.2) converges. But, as shown by Carleman in [10] for series of Stieltjes, Eq. (B.2) may be replaced by

$$\sum \left(\frac{1}{c_n^{1/(2n)}}\right) \text{ diverges,} \tag{B.3}$$

which is the case for $c_n = (2n)!$

What we do, therefore, is to relate the renormalized field theory series to a finite number of series of Stieltjes. The main part of this paper shows how these might actually be constructed. The sum of each of these series of Stieltjes is unique and can be constructed (by means of Padé approximants, for example). Thus, in this sense, a renormalized field theory series has a unique sum which can be constructed.

Without going into full detail, we can indicate some of the ways field theory is related to series of Stieltjes. For example, in order to eliminate disconnected graphs from a scattering amplitude, one divides out the vacuum to vacuum graphs:

$$f_{\text{connected}} = \frac{f_{\text{disconnected}}}{f_0} = \frac{\exp(2i\delta_2)}{\exp(2i\delta_0)}. \tag{B.4}$$

Thus,

$$\tan\delta = \tan(\delta_2 - \delta_0) = \frac{\tan\delta_2 - \tan\delta_0}{1 + \tan\delta_2 \tan\delta_0}. \tag{B.5}$$

One shows that (let us admit this for the moment for sake of argument: what one actually shows is discussed immediately below) $\tan\delta_2$ and $\tan\delta_0$ are series of Stieltjes (when divided by the square of the coupling constant z), so that one encounters

$$f = \frac{U - V}{1 + z^2 UV} \tag{B.6}$$

which is similar to Eq. (1.2) (the factor is z^2 instead of z, and the expansion parameter is a coupling constant squared rather than an energy).

To prove that $\tan\delta_2$ is a series of Stieltjes (or to show how it is related to series of Stieltjes), one considers

$$\tan\delta_2 = -\left\langle \bar{\varphi} \,\middle|\, zV - z^2 V\frac{P}{a}V + z^3 V\frac{P}{a}V\frac{P}{a}V - \cdots \,\middle|\, \varphi \right\rangle, \tag{B.7}$$

where φ is a wave function in some space, z an expansion parameter, V an interaction, P means principal part, and $1/a$ is a propagator, $a = \mathscr{H}_0 - E_0$, $\mathscr{H}_0$ = unperturbed Hamiltonian. If V is one sign definite (say positive), then

$$V = \sqrt{V}\,\sqrt{V}, \tag{B.8}$$

and one has

$$\tan\delta_2 = -z\left\langle\bar{\varphi}\sqrt{V}\,\middle|\,1 - z\sqrt{V}\frac{P}{a}\sqrt{V} + z^2\left(\sqrt{V}\frac{P}{a}\sqrt{V}\right)\left(\sqrt{V}\frac{P}{a}\sqrt{V}\right) - \cdots\,\middle|\,\sqrt{V}\,\varphi\right\rangle.$$

Let

$$K = \sqrt{V}\frac{P}{a}\sqrt{V} \tag{B.9}$$

have eigenfunctions and eigenvalues as follows:

$$K\psi_n = \lambda_n\psi_n\,. \tag{B.10}$$

We assume that K is self-adjoint so that the λ_n's are real. Expand

$$\begin{aligned} \bar{\varphi}\sqrt{V} &= \sum_n c_n{}^*\bar{\psi}_n\,, \\ \sqrt{V}\,\varphi &= \sum_n c_n\psi_n\,. \end{aligned} \tag{B.11}$$

Then

$$\tan\delta_2 = -z\sum_n \frac{c_n{}^*c_n}{1+\lambda_n z} = z\int_{-\infty}^{\infty}\frac{d\Psi(t)}{1+tz}, \tag{B.12}$$

where $\Psi(t)$ is the monotonically increasing step function with discontinuities $c_n{}^*c_n$ at $t = \lambda_n$.

If all the eigenvalues of K are positive (or negative), then the range of integration in Eq. (B.12) is 0 to ∞ (or $-\infty$ to 0), and $\tan\delta/z$ is a series of Stieltjes. [Were the full range necessary, Eq. (B.3) would be

$$\sum_n \left(\frac{1}{c_n^{1/n}}\right) \text{ diverges,}$$

which is not the case for $c_n = (2n)!$; one must relate field theory to strict series of Stieltjes.] If the eigenvalues are bounded from one side (say below), then

$$z\int_{-\infty}^{\infty}\frac{d\Psi(t)}{1+tz} = z\int_0^{\infty}\frac{d\Psi'(t')}{1+(t'-a)\,z} = \frac{z}{1-za}\int_0^{\infty}\frac{d\Psi'(t')}{1+\dfrac{z}{1-za}t'};$$

that is, $\tan \delta_2$ is a series of Stieltjes in the variable $z/(1 - za)$. This fact suggests a problem in the construction of series of Stieltjes to which the method discussed in the main text may be applied.

In case V is not positive definite (even for interactions like $\lambda\varphi^4$ in field theory, the fact that normal ordered products: $\lambda\varphi^4$: actually occur in the theory destroys the seemingly obvious positive definiteness of the interaction—still, the use of normal ordered products only eliminates certain "tadpole" graphs from the theory: one can use $\lambda\varphi^4$ and keep the tadpole graphs: certain divergences will occur which have to be dealt with by regularization—since one has to regularize anyhow it is not clear that this matters at all), we proceed by noting that $P/\mathscr{H}_0 - E_0$ will be positive definite provided $E_0 < 0$. (One argues that the field theory series are unique and can be constructed for negative energies—we are left with the problem of extrapolating to positive energy which is another problem—namely, dispersion theory). In that case we consider

$$\frac{\tan \delta + z \langle \bar{\varphi} \mid V \mid \varphi \rangle}{z^2} = \left\langle \bar{\varphi} V \sqrt{\frac{P}{a}} \mid 1 - zK + z^2K^2 - \cdots \mid \sqrt{\frac{P}{a}}\, V\varphi \right\rangle,$$

where

$$K = \sqrt{\frac{P}{a}}\, V \sqrt{\frac{P}{a}}.$$

Then $[\tan \delta + z\langle \varphi \mid V \mid \varphi \rangle / z^2$ is a series of Stieltjes, or related to a series of Stieltjes, provided that the eigenvalues of K are one sign definite or bounded from one side.

Still another type of problem is suggested by charge renormalization. In electrodynamics, one may consider that the renormalized charge squared $(e_R{}^2)$ is $(\tan \delta/\sqrt{E})$ evaluated at $E = E_0$: that is,

$$e_R{}^2 = e_0{}^2 + E_4 e_0{}^4 + \cdots, \tag{B.14}$$

where e_0 is the unrenormalized charge. At some other energy

$$\frac{\tan \delta}{\sqrt{E}} = T_2 e_0{}^2 + T_4 e_0{}^4 + \cdots. \tag{B.15}$$

One obtains the renormalized series for $\tan \delta/\sqrt{E}$ by inverting Equation (B.14) and substituting the result in Equation (B.15). One shows that the unrenormalized series are series of Stieltjes (for sake of argument again—the actual situation is more complicated as already discussed at length), and so one encounters this problem: given

$$\begin{aligned} f(z) &= S_1(S_2^{-1}(e_R{}^2)) \\ &= f_0 + f_1 z + f_2 z^2 + \cdots, \end{aligned}$$

construct S_1 and S_2. One sees that the method discussed in the main text is applicable.

The arguments given so far are the results of many conversations between the authors and Professor John Nuttall of Texas A & M University. Many obstacles lie in the way of making such a discussion rigorous and completely satisfying. One such obstacle is amply illustrated by the restriction Eq. (A.2). What happens when $c \to \infty$; that is, what happens when the function one is seeking is the limit of a sequence of functions related to series of Stieltjes? An elementary example is provided by

$$f(z) = P \int_0^1 \frac{dt}{(t - \frac{1}{2})(1 + zt)}$$

$$\lim_{\epsilon \to 0} \int_0^1 \frac{(t - \frac{1}{2})}{(t - \frac{1}{2})^2 + \epsilon^2} \frac{dt}{(1 + zt)}$$

$$= \lim_{\epsilon \to 0} (U_\epsilon - V_\epsilon),$$

where U_ϵ and V_ϵ are series of Stieltjes. Since neither $\lim_{\epsilon \to 0} U_\epsilon$ nor $\lim_{\epsilon \to 0} V_\epsilon$ exists, $f(z)$ is *not* the difference of two series of Stieltjes: it is the limit of a sequence of functions which *are* differences of series of Stieltjes. In field theory, one encounters this problem because he will initially work with regulated propagators in order to get finite quantities. He may want to work in a space-time box to avoid delta functions. Ultimately, he will want to pass to a limit in which all regularizing masses and the space-time box are infinite.

Some of the recent progress made by Glimm and Jaffe [12] may provide a better starting point for progress in field theory than the above remarks do. Conceivably, their work will produce a proof that the renormalized series are asymptotic. In that case, contact may be made with another type of result given by Carleman. For example, Carleman shows that there is only one function $f(x)$ satisfying

$$\left| \frac{f(x) - \sum_{=0}^{n-1} c_\nu x^\nu}{x^n} \right| \leqslant m_n$$

provided

$$\sum \frac{1}{\sqrt[n]{m_n}} \text{ diverges.}$$

But at present it seems that the task of obtaining such error estimates are as difficult as the task of overcoming the obstacles in our program.

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